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Non-Gaussian Isocurvature Perturbations From Goldstone Modes Generated During Inflation

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Abstract

We investigate non-Gaussian isocurvature perturbations generated by the evolution of Goldstone modes during inflation. If a global symmetry is broken *before* inflation, the resulting Goldstone modes are disordered during inflation in a precise and predictable way. After inflation these Goldstone modes order themselves in a self-similar way, much as Goldstone modes in field ordering scenarios based on the Kibble mechanism. For $(H_{inf}^2/M_{pl}^2) \sim 10^{-6}$, through their gravitational interaction these Goldstone modes generate density perturbations of approximately the right magnitude to explain the cosmic microwave background (CMB) anisotropy and seed the structure seen in the universe today. We point out that for the pattern of symmetry breaking in which a global $U(1)$ is completely broken, the inflationary evolution of the Goldstone field may be treated as that of a massless scalar field. Unlike the more commonly discussed case in which a global $U(1)$ is completely broken in a cosmological phase transition, in the inflationary case the production of defects can be made exponentially small, so that Goldstone field evolution is completely linear. In such a model non-Gaussian perturbations result because to lowest order density perturbations are sourced by products of Gaussian fields. Consequently, in this non-Gaussian model N-point correlations may be calculated by evaluating Feynman diagrams. We explore the issue of phase dispersion and conclude that this non-Gaussian model predicts Doppler peaks in the CMB anisotropy.

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1. Introduction

A key problem of modern cosmology is the origin of the primordial fluctuations that later led to structure in the universe. From a theoretical perspective, at this point the most satisfying theory for the very early universe is inflationary cosmology because of its nice resolution of the horizon, smoothness, flatness, and monopole problems.[1] In its simplest form, inflation with a single real scalar field predicts an approximately scale-invariant spectrum of Gaussian adiabatic primordial density perturbations.[2] But if other light scalar fields are present during inflation (light relative to the expansion rate H_{inf} during inflation), other types of density perturbations are generated as well, and these other types of perturbations leave a different and distinguishable imprint on the universe today.

The observation that inflation can produce perturbations other than the commonly discussed almost scale-free Gaussian adiabatic density perturbations is not new. Possibilities have been pointed out by various authors. In inflationary models with a Peccei-Quinn symmetry broken before inflation, the inflationary expansion disorders the axion field, which during inflation may be regarded as a massless scalar field. After inflation, when the axion mass becomes relevant, fluctuations in the axion field translate into perturbations in the axion mass density. Allen, Grinstein, and Wise[3] observed that a fourth-order derivative term in the effective theory for the Peccei-Quinn symmetry breaking gives rise to non-Gaussian fluctuations in the axion density. There are other possibilities for non-Gaussianity in axion density perturbations from inflation;[4] however, in many cases inflationary axion perturbations are very nearly Gaussian. It has also been pointed out that for inflation with multiple scalar fields possibilities exist for non-Gaussian fluctuations.[5,6] For example, when the classical slow-roll path bifurcates into two paths toward the minimum, in what may be described as hitting a mogul, quantum fluctuations about the slow-roll path determine which side of the mogul the inflaton field chooses and non-Gaussian fluctuations result. Fluctuations from finite size bubbles in extended inflation[7] give non-Gaussian density perturbations. Also by combining inflation with various sorts of topological defect models non-Gaussian perturbations may be produced.[8,9]

In this paper we investigate the evolution of Goldstone modes during inflation and the density perturbations subsequently generated. We assume an exactly massless Goldstone mode and also that the potential for the inflaton field is such that the usual

adiabatic Gaussian perturbations from quantum fluctuations of the inflaton field are so small as to be observationally irrelevant. We assume that unlike in axion models the global symmetry giving rise to the Goldstone mode remains exact today. The effect of even a small Goldstone mass drastically changes the analysis of the perturbations. Vilenkin in 1982 pointed out how the ordering of an initially disordered Goldstone field can generate density perturbations of the right order to explain the origin of structure in the universe.[10] Vilenkin considered a Goldstone field initially disordered through a symmetry-breaking phase transition, so that on large scale there are no correlations. The evolution of Goldstone modes for large N have been studied by Turok and Spergel.[11] Here we assume that the global symmetry is broken *before* inflation so that the inflationary dynamics of the Goldstone fields determine the state of the Goldstone field on superhorizon scales at the end of inflation.

Müller and Schmid have also considered density fluctuations arising from an inflationary Goldstone mode.[12] Their analysis differs from ours in that they compute only correlations of the Goldstone field contribution to the stress-energy. The cosmological perturbations that we observe today are perturbations in the gravitational potential and in the total stress-energy. Since the contribution of the Goldstone modes to the total stress-energy is small, it is justified to ignore the effect of gravity on the evolution of the Goldstone modes, treating them as a stiff source. The rest of the matter in the universe, which gives the overwhelming contribution to the total stress-energy, may be treated as a single-component perfect fluid, first with the equation of state of radiation and later with a pressureless equation of state. The Goldstone stiff source excites perturbations of this single-component fluid coupled to gravity. Cosmological perturbations observed today are computed using Green functions—their correlations are related by an integral transform to correlations at earlier times of the sort considered by Müller and Schmid.

Inflationary Goldstone modes are interesting for two reasons. First, if we are to confront inflation with observation, we must fully explore the possibilities offered by inflationary cosmology rather than exclusively focusing on the predictions of the simplest models. The second motivation has to do with the simplicity of this model and its qualitative similarity to field ordering models based on the Kibble mechanism (i.e., cosmic strings, global monopoles, textures, nonlinear Goldstone modes, etc.). For inflationary Goldstone modes, as we shall show, in many situations their evolution

is linear, or almost linear. The resulting density fluctuations, however, are non-Gaussian because they are sourced by the stress-energy from the Goldstone modes, which is the sum of products of two Gaussian fields rather than a Gaussian field itself. Because of this, the expectation values of the CMB multipole moments are four-point functions in a Gaussian theory, and thus comparatively simple to evaluate. Despite the underlying Gaussian character of these linear inflationary modes, this model is remarkably similar to field ordering models based on the Kibble mechanism. In both cases one starts (either after inflation or after the symmetry-breaking phase transition) with a disordered order parameter field that orders itself in a self-similar way, with the characteristic length scale at any time equal to the Hubble length at that time. In both cases the matter density perturbations are sourced by a Goldstone mode stress-energy $\Theta_{\mu\nu}(\mathbf{x}, t)$ that is non-Gaussian, so that a simple decomposition into momentum modes is not possible. The principal difference between inflationary Goldstone modes and Goldstone modes from the Kibble mechanism lies in the correlations of the initial field configurations on superhorizon scales. With the Kibble mechanism there are initially no correlations on large scales because of considerations of causality, but in the inflationary case at reheating there exist correlations on arbitrarily large scales. However, this difference is expected to have a small effect, probably only changing the parameters of a scaling solution but not its qualitative behavior.

Recently there has been much discussion in the literature about the structure of the small-angle CMB multipole moments in field ordering theories.[13−17] An important qualitative issue is whether field ordering theories predict oscillations, or *Doppler* peaks, in the large-ℓ CMB multipole moments, similar to those predicted for adiabatic Gaussian fluctuations from inflation. For adiabatic Gaussian fluctuations each mode in momentum space has a definite fixed *phase*, although its real amplitude is a Gaussian random variable. Therefore modes of wavelength comparable to or smaller than the Hubble length at last scattering undergo acoustic oscillations with a definite, fixed phase. By contrast, for non-Gaussian isocurvature models these acoustic modes have at least some dispersion in their phase. Since these modes are sourced primarily during horizon crossing over a time of order the Hubble time, there is no reason to expect what would be the decaying mode at earlier times to be absent, and since the source is nonlinear, there is no reason to expect that the time dependence of the source for a given mode, as opposed to its overall amplitude, not to fluctuate. A fixed

time dependence would eliminate any phase dispersion. Therefore, it is clear that oscillations of the CMB moments in field ordering theories are at least to some extent suppressed, but how much is a detailed quantitative question. While there definitely exist effects that tend to smear the phase, it is not *a priori* clear whether these effects are strong enough to suppress the oscillatory character of the small-angle CMB multipole moments. Crittenden and Turok[14] have found oscillatory Doppler peaks for global textures (where the order parameter space is $SU(2) \cong S^3$). Their model for computing the CMB moments assumes a fixed time dependence for the source for each mode, thus not allowing for any phase dispersion. Durrer et al.[15] have also studied the CMB moments for textures, obtaining results essentially in agreement with Crittenden and Turok. Albrecht et al.,[16] on the other hand, have suggested that phase dispersion may be very significant, especially for cosmic string models, which they claim exhibit no secondary Doppler peaks. It is hoped that the CMB anisotropy moments from linearized inflationary Goldstone modes, which because of its simplicity can be solved exactly in terms of four-point functions in a Gaussian theory, may shed some light on these questions.

Goldstone modes arise through the breaking of a global symmetry G to a smaller group H, giving rise to massless modes equal in number to the difference in dimensionality of the two groups. Most studied has been the evolution of such modes after a cosmological phase transition. In these field ordering or topological defect models, it is typically assumed that initially (either because of preferred initial conditions or because of a prior epoch of inflation) one starts with an exactly homogeneous and isotropic universe at a high temperature in which the symmetry G is unbroken. Later, as the universe cools, a phase transition takes place in which the symmetry G breaks to H. Immediately after the phase transition at each point an orientation of G/H is chosen at random, but beyond some correlation length, which by considerations of causality cannot exceed the horizon size, the orientations are uncorrelated. Subsequently the order parameter field orders itself in a self-similar way, described by a scaling solution, which in general must be determined numerically.

In this paper we consider a situation in which the symmetry G is broken to H *before* inflation. In this case the state of the field G/H at the end of inflation is completely determined by the inflationary dynamics of the field G/H during inflation. All vestiges of initial conditions before inflation are erased. After inflation the coset

field evolves classically, becoming aligned on increasingly larger scales, just as in the scenarios based on the Kibble mechanism. Although the field ordering dynamics in the two cases (after the end of inflation or after the phase transition) are the same, the initially conditions at the respective times are quite different. After the phase transition the coset field is uncorrelated over large distances—one starts with a white noise spectrum on large scales. By contrast, at the end of inflation there exist correlations on arbitrarily large scales.

In the case where $H_{inf} \ll f_g$ (where H_{inf} is the Hubble constant during inflation and f_g is the symmetry breaking scale, roughly the radius of the coset space), one may adopt the following stochastic picture of the inflationary dynamics of the coset field, taken from the stochastic approach to chaotic inflation.[18] We shall ignore fluctuations of the coset field on scales less than the Hubble length H_{inf}^{-1}, treating the coset field as classical and constant over a Hubble volume, a cube of volume H_{inf}^{-3}. As the universe expands by a factor of two, each such cube subdivides into eight subcubes, each of the original size, and in each of these subcubes the coset field takes random step of order $\Delta\phi \approx H_{inf}$. The random steps in different subcubes are uncorrelated. This process repeats until the end of inflation. Physically, the random steps represent the freezing in of quantum fluctuations on subhorizon scales by the inflationary expansion.

In the most general case, owing to nonlinearity, the evolution of the nonlinear sigma model G/H during and after inflation is quite complicated and thus not amenable to an analytic treatment. However in special cases nonlinearity plays a negligible role. When $G/H \cong S^1$, the nontrivial topology of S^1 is irrelevant, and we may replace S^1 by the real line, treating the Goldstone mode as a free massless real scalar field. If the potential in the radial direction is sufficiently stiff, the process by which the field jumps over the origin creating a configuration of nonzero winding number is exponentially suppressed.[9] This process may be described as the nucleation of a cosmic string loop. For the loop not to recollapse, it must have a radius larger than a Hubble length. Note that for this pattern of symmetry breaking, in scenarios based on the Kibble mechanism nonlinearity is always important, because there is no way to suppress the formation of cosmic strings. The other case in which inflationary Goldstone modes may be linearized arises when $H_{inf} \ll f_g$. In this case, provided one does not consider too large a volume, the Goldstone field never wanders far enough from its average value for nonlinear effects to become important.

The organization of this paper is as follows. In section II we present the initial condition for a linearized Goldstone mode at the end of inflation and give its evolution into the radiation-dominated era as well as the resulting contribution of the Goldstone mode to the stress-energy. In section III the resulting stress-energy is renormalized and the effect of this source through gravity on the radiation fluid is computed. A derivation of the Green's functions for computing the perturbations of the radiation fluid as an integral transform over the stiff source stress-energy is relegated to an appendix. Because our computation does not include the decaying modes of the Goldstone field, our calculations become unreliable on subhorizon scales. We follow the evolution of the radiation fluid modes until horizon crossing and compute the phase dispersion in terms of a covariance matrix of two point functions of the two radiation fluid modes. We find the amount of phase dispersion to be small. In Section IV we present some concluding remarks indicating future directions.

2. Goldstone Mode Evolution and Stress-Energy

For simplicity we consider a single Goldstone mode represented by a real massless scalar field, as in the $G/H \cong S^1$ model. The generalization to several Goldstone modes is straightforward. After the end of inflation we may treat this field as a Gaussian ensemble of classical fields that evolve classically, just as one typically assumes in the discussion of the usual adiabatic perturbations from inflation. Initially, on superhorizon scales one has

$$\phi(\mathbf{x}, \eta = 0) = \frac{H_{inf}}{\sqrt{2}} \int \frac{d^3k}{(2\pi)^3}\ k^{-3/2} a(\mathbf{k})\ e^{i\mathbf{k}\cdot\mathbf{x}} \tag{2.1}$$

where H_{inf} is the Hubble constant during inflation and the $a(\mathbf{k})$'s are complex Gaussian random variables subject to the constraint $a(\mathbf{k}) = a(-\mathbf{k})^*$ with the two-point function

$$\langle a(\mathbf{k}) a(\mathbf{k}') \rangle = (2\pi)^3 \cdot \delta^3(\mathbf{k} + \mathbf{k}'). \tag{2.2}$$

The subsequent evolution of the Goldstone modes is given by

$$\phi(\mathbf{x}, \eta) = \frac{H_{inf}}{\sqrt{2}} \int \frac{d^3k}{(2\pi)^3}\ k^{-3/2} a(\mathbf{k})\ e^{i\mathbf{k}\cdot\mathbf{x}}\ T(\eta, k) \tag{2.3}$$

where $T(\eta, k)$ satisfies

$$\ddot{T} + \frac{2\dot{a}(\eta)}{a(\eta)}\dot{T} + k^2 T = 0 \tag{2.4}$$

and the boundary conditions $T(\eta = 0, k) = 1$, $\dot{T}(\eta = 0, k) = 0$. Here the spacetime metric is $ds^2 = a^2(\eta) \cdot [-d\eta^2 + d\mathbf{x}^2]$ and the dots denote derivatives with respect to conformal time η.

The stress-energy from the massless scalar field is given by

$$\Theta_{\mu\nu} = (\partial_\mu \phi)(\partial_\nu \phi) - \frac{1}{2} g_{\mu\nu}[g^{\alpha\beta}(\partial_\alpha \phi)(\partial_\beta \phi)]. \tag{2.5}$$

Assuming a radiation-dominated universe so that $a(\eta) = \eta$, we may solve eqn. (2.4) explicity, obtaining

$$T(\eta; k) = j_0(k\eta) = \frac{\sin[k\eta]}{(k\eta)}. \tag{2.6}$$

Here we have given an explicit analytical solution for the evolution of the stiff source. In most field ordering theories (such as textures, global monopoles, and cosmic strings) the evolution of the stiff source must be computed numerically on a huge lattice.

In computing matrix elements involving $\Theta_{\mu\nu}(\mathbf{k})$ it is necessary to renormalize. In this paper we are concerned with computing two-point functions of the form

$$\langle \Theta_{\mu\nu}(\mathbf{k}) \Theta_{\mu'\nu'}(-\mathbf{k}) \rangle. \tag{2.7}$$

These can be renormalized á la Zimmerman by replacing the unrenormalized integrand $I_{unrenorm}(\mathbf{k})$ with $I_{renorm}(\mathbf{k}) = I_{unrenorm}(\mathbf{k}) - I_{unrenorm}(\mathbf{k} = 0)$. Given the approximations used here, in which the decaying mode is ignored resulting in a description for the behavior of the scalar field accurate only on superhorizon scales, the need to renormalize is not completely manifest, because integrating over the unrenormalized integrands gives convergent integrals, with dominant contribution from momenta of order $k \approx H_{inf}$. But $k \approx H_{inf}$ is precisely where the approximations used here break down. The ultraviolet divergence has been avoided because we have

extrapolated the infrared enhanced correlations on superhorizon scales to subhorizon scales. In a sense through our approximations we have introduced a cutoff at $k_{internal} \approx H_{inf}$. The renormalized matrix elements can be computed straightforwardly by subtracting the integrand at $k = 0$. After this subtraction at any given time η the dominant modes contributing to $\Theta_{\mu\nu}$ are peaked around $k \approx \eta^{-1}$ (i.e., of wavelength of order the horizon size, or Hubble length), as one might expect.

For future reference we give the expansion of the Fourier components

$$\Theta_{\mu\nu}(\mathbf{k}) = \int d^3x \; e^{-i\mathbf{k}\cdot\mathbf{x}} \; \Theta_{\mu\nu}(\mathbf{x}) \tag{2.8}$$

in terms of the Gaussian random variables $a(\mathbf{k})$ and decompose these Fourier components into their *scalar, vector,* and *tensor* parts.

Substituting eqn. (2.3) into eqn. (2.5), one obtains

$$\begin{aligned}\Theta_{00}(\mathbf{k}) = \frac{H_{inf}^2}{2} \int \frac{d^3k'}{(2\pi)^3} \, \frac{a(\mathbf{k}-\mathbf{k}')a(\mathbf{k}')}{|\mathbf{k}-\mathbf{k}'|^{\frac{3}{2}} \; |\mathbf{k}'|^{\frac{3}{2}}} \\ \times \left[\frac{1}{2}\frac{\partial T(\eta,|\mathbf{k}-\mathbf{k}'|)}{\partial\eta}\frac{\partial T(\eta,|\mathbf{k}'|)}{\partial\eta} - \frac{1}{2}(\mathbf{k}-\mathbf{k}')\cdot\mathbf{k}' \; T(\eta,|\mathbf{k}-\mathbf{k}'|) \; T(\eta,|\mathbf{k}'|)\right],\end{aligned} \tag{2.9}$$

$$\begin{aligned}\Theta_{0i}(\mathbf{k}) = \frac{1}{4} H_{inf}^2 \int \frac{d^3k'}{(2\pi)^3} \, \frac{a(\mathbf{k}-\mathbf{k}')a(\mathbf{k}')}{|\mathbf{k}-\mathbf{k}'|^{\frac{3}{2}} \; |\mathbf{k}'|^{\frac{3}{2}}} \\ \times \left[+i \; (\mathbf{k}')_i \frac{\partial T(\eta,|\mathbf{k}-\mathbf{k}'|)}{\partial\eta} \; T(\eta,|\mathbf{k}'|) + i \; (\mathbf{k}-\mathbf{k}')_i \frac{\partial T(\eta,|\mathbf{k}'|)}{\partial\eta} \; T(\eta,|\mathbf{k}-\mathbf{k}'|)\right],\end{aligned} \tag{2.10}$$

$$\begin{aligned}\Theta_{ij}(\mathbf{k}) = \frac{H_{inf}^2}{2} \int \frac{d^3k'}{(2\pi)^3} \, \frac{a(\mathbf{k}-\mathbf{k}')a(\mathbf{k}')}{|\mathbf{k}-\mathbf{k}'|^{\frac{3}{2}} \; |\mathbf{k}'|^{\frac{3}{2}}} \\ \times \left[\frac{1}{2}\frac{\partial T(\eta,|\mathbf{k}-\mathbf{k}'|)}{\partial\eta}\frac{\partial T(\eta,|\mathbf{k}'|)}{\partial\eta} \; \delta_{ij} + \frac{1}{2}(\mathbf{k}-\mathbf{k}')\cdot\mathbf{k}' \; T(\eta,|\mathbf{k}-\mathbf{k}'|) \; T(\eta,|\mathbf{k}'|) \; \delta_{ij} \right. \\ \left. - \, (\mathbf{k}-\mathbf{k}')_i \; (\mathbf{k}')_j \; T(\eta,|\mathbf{k}-\mathbf{k}'|) \; T(\eta,|\mathbf{k}'|)\right].\end{aligned} \tag{2.11}$$

We decompose the spatial-spatial part of $\Theta_{ij}(\mathbf{k})$ as follows:

$$\Theta_{ij}^S(\mathbf{k}) = \delta_{ij} \; \Theta_L(\mathbf{k}) + \left(\frac{k_i k_j - \frac{1}{3}k^2\delta_{ij}}{k^2}\right) \Theta_T(\mathbf{k}), \tag{2.12}$$

so that

$$\Theta_L(k) = \frac{H_{inf}^2}{2} \int \frac{d^3k'}{(2\pi)^3} \; \frac{a(\mathbf{k}-\mathbf{k}')a(\mathbf{k}')}{|\mathbf{k}-\mathbf{k}'|^{\frac{3}{2}} \; |\mathbf{k}'|^{\frac{3}{2}}} \times \left[\frac{1}{2} \frac{\partial T(\eta, |\mathbf{k}-\mathbf{k}'|)}{\partial \eta} \frac{\partial T(\eta, |\mathbf{k}'|)}{\partial \eta} + \frac{1}{6}(\mathbf{k}-\mathbf{k}') \cdot \mathbf{k}' \; T(\eta, |\mathbf{k}-\mathbf{k}'|) \; T(\eta, |\mathbf{k}'|) \; \right] \tag{2.13}$$

and

$$\Theta_T(k) = \frac{H_{inf}^2}{2} \int \frac{d^3k'}{(2\pi)^3} \; \frac{a(\mathbf{k}-\mathbf{k}')a(\mathbf{k}')}{|\mathbf{k}-\mathbf{k}'|^{\frac{3}{2}} \; |\mathbf{k}'|^{\frac{3}{2}}} \times \left[\left(-(\mathbf{k}\cdot\mathbf{k}') + \frac{3}{2}\frac{(\mathbf{k}\cdot\mathbf{k}')^2}{k^2} - \frac{1}{2} \; \mathbf{k}'^{\;2} \right) T(\eta, |\mathbf{k}-\mathbf{k}'|) \; T(\eta, |\mathbf{k}'|) \; \right]. \tag{2.14}$$

3. Scalar Density Perturbations in a Radiation-Dominated Universe

In the previous section we computed the evolution of the stiff source (i.e., the linearized Goldstone modes) and the stress-energy source $\Theta_{\mu\nu}$ resulting from these linearized Golstone modes. In this section we compute the *scalar* density perturbations generated by this source, relegating many of the computational details and a precise description of the notational conventions to Appendix A.

For computational simplicity, to make the problem as scale free as possible, we assume a radiation-dominated universe, with $a(\eta) = \eta$ and a single-component fluid with $c_s^2 = \frac{1}{3}$ and work in momentum space. It follows using the Green functions derived in Appendix A that for the Newtonian potential

$$\psi(\mathbf{k}, \eta) = \eta^{-1} \cdot \left[\; \psi(\mathbf{k}, j, \eta) \; j_1(k\eta/\sqrt{3}) + \psi(\mathbf{k}, y, \eta) \; y_1(k\eta/\sqrt{3}) \; \right], \tag{3.1}$$

where

$$\psi(\mathbf{k}, \eta, j) = -(4\pi G) \; \left(\frac{k}{\sqrt{3}} \right) \int_0^\eta d\bar{\eta} \; \bar{\eta}^3 \; y_1(k\bar{\eta}/\sqrt{3}) \times \left[\Theta_L(\mathbf{k}, \bar{\eta}) - \frac{1}{3} \; \Theta_{00}(\mathbf{k}, \bar{\eta}) + \frac{2}{3}\Theta_T(\mathbf{k}, \bar{\eta}) - \frac{2\dot{\Theta}_T(\mathbf{k}, \bar{\eta})}{k^2\bar{\eta}} \right] \tag{3.2}$$

and similarly

$$\psi(\mathbf{k},\eta,y) = +(4\pi G)\ \left(\frac{k}{\sqrt{3}}\right) \int\limits_0^{\eta} d\bar\eta\ \bar\eta^3\ j_1(k\bar\eta/\sqrt{3}) \times \left[\Theta_L(\mathbf{k},\bar\eta) - \frac{1}{3}\Theta_{00}(\mathbf{k},\bar\eta) + \frac{2}{3}\Theta_T(\mathbf{k},\bar\eta) - \frac{2\dot\Theta_T(\mathbf{k},\bar\eta)}{k^2\bar\eta}\right]. \tag{3.3}$$

We now formulate in more quantitative terms the issue of phase dispersion discussed in the introduction. We compute the covariance matrix

$$\mathrm{Cov}(\mathbf{k},\eta) = \begin{pmatrix} \langle\psi(\mathbf{k},\eta,j)\ \psi(-\mathbf{k},\eta,j)\rangle & \langle\psi(\mathbf{k},\eta,j)\ \psi(-\mathbf{k},\eta,y)\rangle \\ \langle\psi(\mathbf{k},\eta,y)\ \psi(-\mathbf{k},\eta,j)\rangle & \langle\psi(\mathbf{k},\eta,y)\ \psi(-\mathbf{k},\eta,y)\rangle \end{pmatrix}. \tag{3.4}$$

Here j and y refer to the two types of Bessel functions—or equivalently, the growing and decaying modes, respectively.

The eigenvalues of the covariance matrix provide a measure of the phase dispersion. For the usual adiabatic perturbations from inflation (for which only a growing mode is present), the covariance matrix is proportional to $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. In the most general case with no phase dispersion, this matrix has one vanishing eigenvalue, indicating a definite fixed linear relationship between $\psi(\mathbf{k},j)$ and $\psi(\mathbf{k},y)$. By contrast, a completely random phase gives a covariance matrix proportional to $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

We now compute the matrix element

$$
\begin{aligned}
&\langle \psi(\mathbf{k},\eta,y)\ \psi(-\mathbf{k}',\eta',y)\rangle \\
&= H_{inf}^4 (4\pi G)^2 \frac{kk'}{12} \cdot \int \frac{d^3k_1}{(2\pi)^3} \int \frac{d^3k_2}{(2\pi)^3} \int_0^{\eta} d\eta_1\ \eta_1^3 \int_0^{\eta'} d\eta_2\ \eta_2^3\ j_1(k\eta_1/\sqrt{3})\ j_1(k\eta_2/\sqrt{3}) \\
&\times \frac{\langle \hat{a}(\mathbf{k}-\mathbf{k}_1)\ \hat{a}(\mathbf{k}_1)\ \hat{a}(-\mathbf{k}'+\mathbf{k}_2)\ \hat{a}(-\mathbf{k}_2)\rangle}{|\mathbf{k}-\mathbf{k}_1|^{3/2}|\mathbf{k}_1|^{3/2}|\mathbf{k}'-\mathbf{k}_2|^{3/2}|\mathbf{k}_2|^{3/2}} \\
&\times \Bigg[\frac{1}{3}\frac{\partial j_0(|\mathbf{k}-\mathbf{k}_1|\eta_1)}{\partial \eta_1}\frac{\partial j_0(k_1\eta_1)}{\partial \eta_1} \\
&\quad + \left\{\frac{(\mathbf{k}\cdot\mathbf{k}_1)^2}{k^2} - \frac{(\mathbf{k}\cdot\mathbf{k}_1)}{3} - \frac{2}{3}k_1^2\right\} j_0(|\mathbf{k}-\mathbf{k}_1|\eta_1)\ j_0(k_1\eta_1) \\
&\quad\quad + \frac{2}{k^2}\left\{\frac{1}{2}k_1^2 + (\mathbf{k}\cdot\mathbf{k}_1) - \frac{3}{2}\frac{(\mathbf{k}\cdot\mathbf{k}_1)^2}{k^2}\right\}\frac{1}{\eta_1}\frac{\partial}{\partial \eta_1}\Big\{j_0(|\mathbf{k}-\mathbf{k}_1|\eta_1)\ j_0(k_1\eta_1)\Big\}\Bigg] \\
&\times \Bigg[\frac{1}{3}\frac{\partial j_0(|\mathbf{k}'-\mathbf{k}_2|\eta_2)}{\partial \eta_2}\frac{\partial j_0(k_2\eta_2)}{\partial \eta_2} \\
&\quad + \left\{\frac{(\mathbf{k}'\cdot\mathbf{k}_2)^2}{k'^2} - \frac{(\mathbf{k}'\cdot\mathbf{k}_2)}{3} - \frac{2}{3}k_2^2\right\} j_0(|\mathbf{k}'-\mathbf{k}_2|\eta_2)\ j_0(k_2\eta_2) \\
&\quad\quad + \frac{2}{k'^2}\left\{\frac{1}{2}k_2^2 + (\mathbf{k}'\cdot\mathbf{k}_2) - \frac{3}{2}\frac{(\mathbf{k}'\cdot\mathbf{k}_2)^2}{k'^2}\right\}\frac{1}{\eta_2}\frac{\partial}{\partial \eta_2}\Big\{j_0(|\mathbf{k}'-\mathbf{k}_2|\eta_2)\ j_0(k_2\eta_2)\Big\}\Bigg] \\
&\quad - \Bigg(\text{integrand with } \Theta_{\mu\nu}(\mathbf{k}) \to \Theta_{\mu\nu}(\mathbf{k}=0),\ \Theta_{\mu\nu}(\mathbf{k}') \to \Theta_{\mu\nu}(\mathbf{k}'=0)\Bigg).
\end{aligned} \tag{3.5}
$$

The matrix element

$$\langle a(\mathbf{k}-\mathbf{k}_1)\ a(\mathbf{k}_1)\ a(-\mathbf{k}'+\mathbf{k}_2)\ a(-\mathbf{k}_2)\rangle \tag{3.6}$$

as a consequence of Wick's theorem may be decomposed into a sum of products of two-point functions with three terms, one of which vanishes, specifically the one that would result from computing $\langle\psi(\mathbf{k},\eta,j)\rangle\ \ \langle\psi(-\mathbf{k}',\eta,j)\rangle$. Therefore eqn. (3.6) equals

$$(2\pi)^6 \cdot \delta^3(\mathbf{k}-\mathbf{k}') \cdot \Big[\delta^3(\mathbf{k}_1-\mathbf{k}_2) + \delta^3(\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2)\Big], \tag{3.7}$$

allowing us to set $\mathbf{k}=\mathbf{k}'$ and to consider two cases: (1) $\mathbf{k}_2=\mathbf{k}_1$, and (2) $\mathbf{k}_2 = \mathbf{k}-\mathbf{k}_1$. By a simple symmetry the two cases give the same contribution, so that eqn.

(3.5) becomes

$$
\begin{aligned}
& M_{yy}(k;\eta_1,\eta_2)\cdot\delta^3(\mathbf{k}-\mathbf{k}') \\
& = H_{inf}^4(4\pi G)^2\frac{k^2}{12}\cdot\delta^3(\mathbf{k}-\mathbf{k}')\ \int d^3k_1 \int_0^{\eta} d\eta_1\ \eta_1^3 \int_0^{\eta'} d\eta_2\ \eta_2^3\ \frac{j_1(k\eta_1/\sqrt{3})\ j_1(k\eta_2/\sqrt{3})}{|\mathbf{k}-\mathbf{k}_1|^3|\mathbf{k}_1|^3} \\
& \times\Bigg[\frac{1}{3}\frac{\partial j_0(|\mathbf{k}-\mathbf{k}_1|\eta_1)}{\partial\eta_1}\frac{\partial j_0(k_1\eta_1)}{\partial\eta_1} \\
& \quad +\left\{\frac{(\mathbf{k}\cdot\mathbf{k}_1)^2}{k^2}-\frac{\mathbf{k}\cdot\mathbf{k}_1}{3}-\frac{2}{3}k_1^2\right\} j_0(|\mathbf{k}-\mathbf{k}_1|\eta_1)\ j_0(k_1\eta_1) \\
& \quad\quad +\frac{2}{k^2}\left\{\frac{1}{2}k_1^2+(\mathbf{k}\cdot\mathbf{k}_1)-\frac{3}{2}\frac{(\mathbf{k}\cdot\mathbf{k}_1)^2}{k^2}\right\}\frac{1}{\eta_1}\frac{\partial}{\partial\eta_1}\left\{j_0(|\mathbf{k}-\mathbf{k}_1|\eta_1)\ j_0(k_1\eta_1)\right\}\Bigg] \\
& \times\Bigg[\frac{1}{3}\frac{\partial j_0(|\mathbf{k}-\mathbf{k}_1|\eta_2)}{\partial\eta_2}\frac{\partial j_0(k_1\eta_2)}{\partial\eta_2} \\
& \quad +\left\{\frac{(\mathbf{k}\cdot\mathbf{k}_1)^2}{k^2}-\frac{\mathbf{k}\cdot\mathbf{k}_1}{3}-\frac{2}{3}k_1^2\right\} j_0(|\mathbf{k}'-\mathbf{k}_1|\eta_2)\ j_0(k_1\eta_2) \\
& \quad\quad +\frac{2}{k^2}\left\{\frac{1}{2}k_1^2+(\mathbf{k}\cdot\mathbf{k}_1)-\frac{3}{2}\frac{(\mathbf{k}\cdot\mathbf{k}_1)^2}{k^2}\right\}\frac{1}{\eta_2}\frac{\partial}{\partial\eta_2}\left\{j_0(|\mathbf{k}-\mathbf{k}_1|\eta_2)\ j_0(k_1\eta_2)\right\}\Bigg] \\
& \quad -\Bigg(\text{integrand with }\Theta_{\mu\nu}(\mathbf{k})\to\Theta_{\mu\nu}(\mathbf{k}=0)\Bigg).
\end{aligned}
\tag{3.8}
$$

In order to compare the matrix elements in the most meaningful way, we consider the quantities

$$
\begin{aligned}
F_{jj}(k,\eta) &= k^3\cdot\left[\frac{j_1(k\eta/\sqrt{3})}{\eta}\cdot\frac{j_1(k\eta/\sqrt{3})}{\eta}\right]\cdot M_{jj}(k;\eta,\eta),\\
F_{jy}(k,\eta) &= k^3\cdot\left[\frac{j_1(k\eta/\sqrt{3})}{\eta}\cdot\frac{y_1(k\eta/\sqrt{3})}{\eta}\right]\cdot M_{jy}(k;\eta,\eta),\\
F_{yy}(k,\eta) &= k^3\cdot\left[\frac{y_1(k\eta/\sqrt{3})}{\eta}\cdot\frac{y_1(k\eta/\sqrt{3})}{\eta}\right]\cdot M_{yy}(k;\eta,\eta).
\end{aligned}
\tag{3.9}
$$

In Figure 1 the F matrix elements are plotted for various k at $\eta=1$. For $k\eta\gtrsim 1$ the decaying modes of the Goldstone field become important, and our calculation here

becomes unreliable because of the neglect of these modes. [Because of the scaling properties of a radiation-dominated universe, any equal time matrix elements may be obtained by a trivial rescaling of equal time matrix elements at $\eta = 1$.] The k^3 factor has been introduced to make F scale free—for a Harrison-Zeldovich spectrum F would be independent of k. Instead we oberve a linear dependence with k in the F matrix elements, reflecting the fact that the radiation fluid/scalar gravity modes are excited by the Goldstone field source as horizon crossing is approached. On superhorizon scales, as $k \to 0$, the radiation fluid modes are unexcited.

The degree of phase dispersion is determined from ratio of the eigenvalues of the matrix $F = \begin{pmatrix} F_{jj} & F_{jy} \\ F_{yj} & F_{yy} \end{pmatrix}$. For $(k\eta) \lesssim 1$, where our calculation is reliable, F is roughly proportional to $\begin{pmatrix} 1.0 & 0.17 \\ 0.17 & 0.05 \end{pmatrix}$. Because of the dominance of the F_{jj} matrix element, the eigenvectors very nearly coincide with the 'growing' and 'decaying' modes $j_1(k\eta/\sqrt{3})/\eta$ and $y_1(k\eta/\sqrt{3})/\eta$, respectively, with the 'growing' eigenvalue dominating over the 'decaying' eigenvalue by a factor of approximately 49. Thus the amplitude of the other mode is approximately 15% that of the dominant mode, which almost coincides with the growing mode with very little mixing.

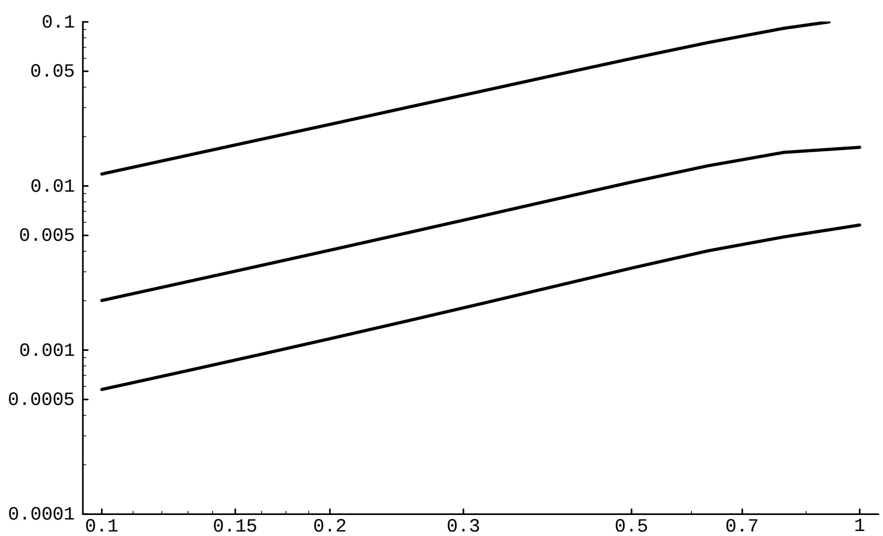

Figure 1. The plotted curves (from top to bottom) represent the matrix elements F_{jj}, F_{jy}, and F_{yy} at $\eta = 1$ as a function of wavenumber k.

4. Discussion

We have presented a class of plausible inflationary models with Goldstone modes that generate non-Gaussian density perturbations that are comparatively easy to compute. It is of great interest to determine whether the primordial density perturbations of our universe inferred from observation are Gaussian, as predicted by the simplest inflationary models, or whether they are of a more complicated, non-Gaussian variety. Since Gaussian models comprise an infinitesimal fraction of all possible cosmological models, in order to define meaningful tests of Gaussianity to consider such tests in the abstract is insufficient. Instead proposed tests of Gaussianity must be judged on their ability to rule out plausible non-Gaussian cosmological models.

In all non-Gaussian models based on symmetry-breaking phase transitions—the so-called topological defect models—nonlinearity plays a key role, making the predictions of such models difficult to compute. By contrast, as we have shown, in many cases inflationary Goldstone modes may be linearized with very little error, making the computation of N-point density perturbations a matter of evaluating Feynman diagrams.

One shortcoming of the calculations in this paper, one which we plan to remedy in future work, is that we have retained only the Goldstone modes corresponding to growing modes on superhorizon scales. Because of the neglect of the decaying modes of the underlying Goldstone field, the calculations are reliable only on superhorizon scales and cannot be trusted on subhorizon scales. To extend the calculation to include the decaying modes should pose no real difficulty, and with this extension it should be possible to compute all sorts of correlation functions by evaluating Feynman diagrams. One should be able to compute the CMB moments in this way and higher-order correlation functions, such as the three-point function and beyond. Another future direction involves making sky maps of the CMB anisotropy, a problem for which computing N-point functions would not be practical and for which some sort of nonperturbative renormalization in position space would be required.

Within the limitations due to neglecting the decaying modes, we were able to determine that, somewhat surprisingly, there is very little phase dispersion. The absence of significant phase dispersion suggests that the CMB moments from the inflationary linearized Goldstone mode theory should exhibit a series of well-defined Doppler peaks.

After this work was completed, we became aware of an interesting related paper[20] that discusses non-Gaussian perturbations from inflation generated by a massive scalar field.

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APPENDIX A—Green's Functions for a Stiff Source

We work in Newtonian gauge (which is equivalent to the gauge invariant formalism)[19] and for the stress-energy in addition to the stiff source described by the divergenceless tensor $\Theta_{\mu\nu}$ assume a single-component perfect fluid with sound speed c_s. The line element is $ds^2 = a^2(\eta)\cdot[\eta_{\mu\nu}+h_{\mu\nu}]dx^\mu\, dx^\nu$ and $\eta_{\mu\nu} = \mathrm{diag}[-1,+1,+1,+1]$.

For *scalar* perturbations, $h_{00} = -2\phi$, $h_{0i} = 0$, and $h_{ij} = -2\psi\ \delta_{ij}$, and the linearized scalar Einstein equations read as follows:

$$
\begin{aligned}
\delta G^0_0 &= \frac{-2}{a^2}\cdot[\nabla^2\psi - 3\mathcal{H}(\dot\psi + \mathcal{H}\phi)] \\
&= (8\pi G)\cdot[-\rho_c\ \delta + \Theta^0_0], && (A1a) \\
\delta G^0_i &= \frac{-2}{a^2}\cdot[\dot\psi + \mathcal{H}\phi]_{|i} \\
&= (8\pi G)\cdot[\rho_c(1+w)v_{|i} + \Theta^0_i], && (A1b) \\
\delta G^j_i(L) &= \frac{2}{a^2}\cdot[\ddot\psi + 2\mathcal{H}\dot\psi + \mathcal{H}\dot\phi + (2\dot{\mathcal{H}} + \mathcal{H}^2)\phi + \frac{1}{3}\nabla^2(\phi-\psi)]\ \delta^j_i \\
&= (8\pi G)\cdot[\rho_c c_s^2\ \delta\ \delta^j_i + \Theta^j_i(L)], && (A1c) \\
\delta G^j_i(T) &= \frac{-1}{a^2}\cdot[(\nabla_i\nabla^j - \frac{1}{3}\delta^j_i\ \nabla^2)(\phi-\psi)] \\
&= (8\pi G)\cdot\Theta^j_i(T) && (A1d)
\end{aligned}
$$

where (L) and (T) denote the longitudinal and transverse *scalar* parts of the spatial-spatial tensors, respectively. Here the dots indicate derivatives with respect to conformal time, $\mathcal{H} = (\dot a/a)$, $\rho_c = (3/8\pi G)\mathcal{H}^2 a^{-2}$, and $v_i = v_{|i}$ (i.e., v is the potential for the *scalar* part of the velocity field).

By adding eqn. (A1a) multiplied by c_s^2 to eqn. (A1c), one obtains

$$\begin{aligned}\ddot{\psi} + (2\mathcal{H} + 3\mathcal{H}c_s^2)\dot{\psi} + \mathcal{H}\dot{\phi} + (2\dot{\mathcal{H}} + \mathcal{H}^2 + 3\mathcal{H}^2 c_s^2)\phi + \frac{1}{3}\nabla^2(\phi - \psi) - c_s^2\nabla^2\psi \\ = (4\pi G)\cdot[\Theta_L - c_s^2\Theta_{00}].\end{aligned} \tag{A2}$$

Here $\Theta_{ij}(L) = \delta_{ij}\Theta_L$. Working in momentum space and using eqn. (A1d) to eliminate ϕ in favor of ψ, we may write

$$\begin{aligned}&\ddot{\psi}(\mathbf{k}) + 3\mathcal{H}(1 + c_s^2)\ \dot{\psi}(\mathbf{k}) + (2\dot{\mathcal{H}} + \mathcal{H}^2 + 3\mathcal{H}^2 c_s^2)\ \psi(\mathbf{k}) + c_s^2 k^2\psi(\mathbf{k}) \\ &\quad = (4\pi G)\cdot\Bigg[\Theta_L(\mathbf{k}) - c_s^2\Theta_{00}(\mathbf{k}) \\ &\qquad\qquad + \frac{2}{3}\Theta_T(\mathbf{k}) - \frac{2\mathcal{H}}{k^2}\ \dot{\Theta}_T(\mathbf{k}) - \frac{4\dot{\mathcal{H}} + 2\mathcal{H}^2(1 + 3c_s^2)}{k^2}\ \Theta_T(\mathbf{k})\Bigg].\end{aligned} \tag{A3}$$

Here we have adopted the convention

$$\Theta_{ij}(T)(\mathbf{k}) = \left(\frac{k_i k_j - \frac{1}{3}k^2\delta_{ij}}{k^2}\right)\Theta_T(\mathbf{k}). \tag{A4}$$

For the special case of a completely radiation-dominated universe, so that $a(\eta) = \eta$ and $c_s^2 = \frac{1}{3}$, eqn. (A3) becomes

$$\begin{aligned}&\ddot{\psi}(\mathbf{k}) + \frac{4}{\eta}\dot{\psi}(\mathbf{k}) + \frac{k^2}{3}\psi(\mathbf{k}) \\ &\quad = (4\pi G)\cdot\left[\Theta_L(\mathbf{k}) - \frac{1}{3}\Theta_{00}(\mathbf{k}) + \frac{2}{3}\Theta_T(\mathbf{k}) - \frac{2}{k^2\eta}\dot{\Theta}_T(\mathbf{k})\right].\end{aligned} \tag{A5}$$

which is essentially a Bessel equation of order $\nu = \frac{3}{2}$. The homogeneous version of this equation has the general solution $\psi(\mathbf{k},\eta) = \eta^{-1}\big[\psi(\mathbf{k},j)\ j_1(k\eta/\sqrt{3}) + \psi(\mathbf{k},y)\ y_1(k\eta/\sqrt{3})\big]$. From this homogeneous solution we may construct Green functions, and eqn. (A5) is solved by

$$\psi(\mathbf{k},\eta) = \eta^{-1}\big[\psi(\mathbf{k},\eta,j)j_1(k\eta/\sqrt{3}) + \psi(\mathbf{k},\eta,y)y_1(k\eta/\sqrt{3})\big] \tag{A6}$$

where $j_1(x) = \sin x/x^2 - \cos x/x,\ y_1(x) = -\cos x/x^2 - \sin x/x$,

$$\begin{aligned}\psi(\mathbf{k},\eta,j) = \frac{-k}{\sqrt{3}} \int_0^\eta d\bar{\eta}\ \bar{\eta}^3\ y_1(k\bar{\eta}/\sqrt{3}) \\ \times (4\pi G) \cdot \left[\Theta_L(\mathbf{k},\bar{\eta}) - \frac{1}{3}\Theta_{00}(\mathbf{k},\bar{\eta}) + \frac{2}{3}\Theta_T(\mathbf{k},\bar{\eta}) - \frac{2}{k^2\bar{\eta}}\dot{\Theta}_T(\mathbf{k},\bar{\eta})\right]\end{aligned} \tag{A7a}$$

and

$$\begin{aligned}\psi(\mathbf{k},\eta,y) = \frac{+k}{\sqrt{3}} \int_0^\eta d\bar{\eta}\ \bar{\eta}^3\ j_1(k\bar{\eta}/\sqrt{3}) \\ \times (4\pi G) \cdot \left[\Theta_L(\mathbf{k},\bar{\eta}) - \frac{1}{3}\Theta_{00}(\mathbf{k},\bar{\eta}) + \frac{2}{3}\Theta_T(\mathbf{k},\bar{\eta}) - \frac{2}{k^2\bar{\eta}}\dot{\Theta}_T(\mathbf{k},\bar{\eta})\right].\end{aligned} \tag{A7b}$$